Proposed Implementation of “Non-Physical” Four-Dimensional Polarization Rotations

Gunnar Björk, Katarina Stensson, and Magnus Karlsson, *Senior Member, IEEE, Fellow, OSA*

***Abstract*—Recently one of us proposed a new formalism for modeling electromagnetic wave transformations for coherent communication using a real, four-vector description instead of the conventionally used Jones calculus or the Mueller matrices. The four-vector can then handle all superpositions of two orthogonal polarization basis and two orthogonal time bases (e.g., the in-phase and quadrature phase). In developing this formulation it was found that to provide a general but minimal framework for such rotations, it is natural to divide the six generators of four-dimensional (4d) rotations into two groups of three generators, the right- and the left-isoclinic matrices. Of the six transformations these generators define, it was furthermore found that four of them are readily implemented by linear optical components, while two of then were impossible to implement by such means. In this paper, we detail the reason these two “unphysical” rotations cannot be implemented with linear optics. We also suggest how they can be implemented, but at a cost in the signal-to-noise ratio, and give this minimum cost.**

***Index Terms*—Coherent optical transmission, four-dimensional modulation, optical polarization, quantum noise.**

I. Introduction

Modern long-haul fiber-optic transmission links are based on coherent transmission techniques that are a sophisticated combination of optics and electronics. Especially the coherent receiver depends heavily on digital signal processing (DSP) to recover the polarization and absolute phase of the transmitted signal, which is often referred to as *intradyne* detection [1]. Typically in these systems, binary phase-shift keying is transmitted in parallel in all four quadratures, or equivalently, independent quadrature phase shift keying (QPSK) is transmitted in each polarization component. This was originally demonstrated with a full online DSP implementation by Sun *et al.* in 2008 [2]. The signaling space is then four-dimensional (4d), as explored early by Betti *et al.* [3], and more recently in [4], where constellation optimization and modulation format performances were studied.

The most common channel model of modern coherent links is to use two-dimensional (2d) complex vectors to describe the input/output signal vectors of the channel and a complex 2×2 matrix to relate these vectors. This is the so-called *Jones calculus*, pioneered by Jones *et al.* in a series of papers in the 1940s for the study of optical polarization effects [5], [6].

Alternatively one may use a 4d real space to describe the coherent channel, and this is actually more powerful, as the 4d transformations is a richer set than the complex 2d Jones matrices. The connection and differences between the two descriptions were recently described in some detail in [7] as was mentioned in the abstract. For example, the Jones formalism cannot correct for transmitter and receiver imperfections related to power and/or phase errors between the in-phase and quadrature signals. Thus the real 4d formalism is often preferred in practical DSP implementations [8]. Recently, new DSP algorithms have been proposed that perform simultaneous phase- and polarization tracking in the 4d space [9], [10].

In 4d, lossless transmission is described by a 4d *rotation* matrix, that has six degrees of freedom, in contrast with the unitary Jones matrix which has four. In [7] these two additional rotations were referred to as “unphysical” in the sense that they do not preserve the boson commutation relations in contrast to the other, conventional, operations. However, as we will discuss in this paper, this does not preclude their realization, but their realization is accompanied with added noise, i.e., a signal to noise ratio (SNR) penalty. These operations are similar to, e.g., quantum mechanical cloning [11]–[13], or linear, phase-insensitive amplification [14] in that unitarity and non-commutation, respectively, prevent them from being performed perfectly. Thus there is a limit to how similar to the original one can clone an unknown state [15]. In the same vein, there is an added noise penalty to pay when amplifying an unknown state [14], [16]. It is also not possible to measure, perfectly, the two quadrature components of a field. Any attempt of doing so will add noise to the measurement [17]. This said, nothing prevents one from measuring either quadrature with arbitrary accuracy and precision [18]. The aim of this paper is to quantify the penalty associated with these “unpysical” rotations and, to achieve this goal, propose physical setups for their realization.

II. Setting the Stage

The starting point of the four-dimensional rotation formalism is that classically, a transverse, electromagnetic field under the slowly varying envelope approximation can be expressed as the real four-vector

$$\overline{E} = \begin{pmatrix} \mathrm{Re}(e_x) \\ \mathrm{Im}(e_x) \\ \mathrm{Re}(e_y) \\ \mathrm{Im}(e_y) \end{pmatrix}, \qquad (1)$$

Manuscript received February 05, 2016; revised April 26, 2016; accepted May 26, 2016. Date of publication June 06, 2016; date of current version June 22, 2016. The work of G. Björk and K. Stensson was supported by the Swedish Research Council (VR) through its support of Linnaeus Center of Excellence ADOPT and under Grant 2014-7869-110765-32. The work of M. Karlsson was supported by the Swedish Research Council (VR) under Grant 2015-04239.

G. Björk, and K. Stensson are with the Department of Applied Physics, Royal Institute of Technology, AlbaNova University Center, SE-106 91 Stockholm, Sweden (e-mail: gbjork@kth.se; stenss@kth.se).

M. Karlsson is with the Photonics Laboratory, Department of Microtechnology and Nanoscience, Chalmers University of Technology, SE-412 96 Gothenburg, Sweden (e-mail: magnus.karlsson@chalmers.se).

Digital Object Identifier 10.1109/JLT.2016.2574912

where e_x and e_y are the two linearly polarized and slowly time evolving E-field components along the respective axis, and where the (rapid) time evolution $\exp(i\omega t)$ of a carrier wave has been assumed, but is suppressed in the notation. In a quantum mechanical description it would be natural to work in the Heisenberg picture, where we, if we here also work in a rotating frame, can write

$$\hat{E} = \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{b}_1 \\ \hat{b}_2 \end{pmatrix}, \tag{2}$$

where $\hat{a}_1, \ldots, \hat{b}_2$ are the Hermitian, electric field in-phase and quadrature-phase operators [19]. In analogy with a classical, complex, electric field E, where $Re(E) = (E + E^*)/2$ and $Im(E) = (E - E^*)/(2i)$ they are defined $\hat{a}_1 = (\hat{a} + \hat{a}^\dagger)/2$ and $\hat{a}_2 = (\hat{a} - \hat{a}^\dagger)/(2i)$, and similarly for $\hat{b} = \hat{b}_1 + i\hat{b}_2$. The operator $\hat{a}$ ($\hat{a}^\dagger$) is the non-Hermitian annihilation (creation) operator, and just as E is a complex number and therefore is not directly measurable, $\hat{a}$ and ($\hat{a}^\dagger$) are non-Hermitian operators and therefore not directly measurable either. However, the quadrature electrical fields are physical, measurable quantities in both descriptions.

The operators $\hat{a}_1$ and $\hat{a}_2$ are normalized such that the energy in the mode is $\hbar\omega(\hat{a}_1^2 + \hat{a}_2^2)$. The quantum mechanical expression makes the distinction between the temporal and the polarization degrees of freedom more succinct. While any of the a operators associated with the x-polarized E-field component commutes with any of the b operators associated with the y-polarized components, the $\hat{a}_1$ and $\hat{a}_2$ operators don't commute. From the standard bosonic commutation relation $[\hat{a}, \hat{a}^\dagger] \equiv \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$ and the defining relations above, one can straightforwardly derive the commutation relation [19]

$$[\hat{a}_1, \hat{a}_2] = \hat{a}_1\hat{a}_2 - \hat{a}_2\hat{a}_1 = \frac{i}{2}. \tag{3}$$

Relation (3) implies that the two quadratures cannot be measured simultaneously without penalty. (Similarly the $\hat{b}_1$ and $\hat{b}_2$ operators don't commute.) Classically, all four vector components are "mode orthogonal", either in polarization, in time, or in both, so that a measurement of any combination of them can be done without any "penalty". This is not the case in the quantum mechanical description, and this influences what can be, and what cannot be done to the vector $\hat{E}$.

A consequence of the non-commutability of the quadrature amplitude operators is that they must have a minimum uncertainty product. In this particular case the uncertainty relation reads

$$\begin{aligned} \langle \hat{a}_1 - \langle \hat{a}_1 \rangle \rangle^2 \langle \hat{a}_2 - \langle \hat{a}_2 \rangle \rangle^2 &\equiv (\Delta\hat{a}_1)^2 (\Delta\hat{a}_2)^2 \\ &\geq \frac{|\langle [\hat{a}_1, \hat{a}_2] \rangle|^2}{4} = \frac{1}{16} \end{aligned} \tag{4}$$

according to (3). The analogous relation of course holds for the b operators. If the state of the field is in a coherent state, or in the vacuum state (also called zero-point field), then the uncertainty relation above is satisfied with equality, and moreover the two quadrature amplitude fluctuations are equal. Hence, for such fields $(\Delta\hat{a}_1)^2 = (\Delta\hat{a}_2)^2 = 1/4$ and similar for any other mode in such states.

III. The Rotation Generators and Their Associated Rotation Matrices

In [7] it was found that one complete but minimal set of generators of four-vector rotations is given by the six matrices ρ_1 - ρ_3 and λ_1 - λ_3 below. All six generator matrices are needed since a vector-length preserving rotation in 4d real space has six degrees of freedom. When applied to a field these generators give rise to rotations R_j, where $j = 1, \ldots, 6$. Under each generation matrix the resulting rotation R_j is written. Here, e.g., $R_1 = \exp(\alpha\rho_1)$ and $R_6 = \exp(\alpha\lambda_3)$. Any 4d rotation can thus be written as the exponential of a real coefficients, linear combination of ρ_1 - ρ_3 and λ_1 - λ_3

$$\rho_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$R_1 = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & 0 & -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \tag{5}$$

$$\rho_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$R_2 = \begin{pmatrix} \cos(\alpha) & 0 & 0 & -\sin(\alpha) \\ 0 & \cos(\alpha) & \sin(\alpha) & 0 \\ 0 & -\sin(\alpha) & \cos(\alpha) & 0 \\ \sin(\alpha) & 0 & 0 & \cos(\alpha) \end{pmatrix} \tag{6}$$

$$\rho_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} \cos(\alpha) & 0 & \sin(\alpha) & 0 \\ 0 & \cos(\alpha) & 0 & \sin(\alpha) \\ -\sin(\alpha) & 0 & \cos(\alpha) & 0 \\ 0 & -\sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix} \tag{7}$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$R_4 = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & 0 & -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \quad (8)$$

$$\lambda_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$R_5 = \begin{pmatrix} \cos(\alpha) & 0 & 0 & -\sin(\alpha) \\ 0 & \cos(\alpha) & -\sin(\alpha) & 0 \\ 0 & \sin(\alpha) & \cos(\alpha) & 0 \\ \sin(\alpha) & 0 & 0 & \cos(\alpha) \end{pmatrix} \quad (9)$$

$$\lambda_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and}$$

$$R_6 = \begin{pmatrix} \cos(\alpha) & 0 & \sin(\alpha) & 0 \\ 0 & \cos(\alpha) & 0 & -\sin(\alpha) \\ -\sin(\alpha) & 0 & \cos(\alpha) & 0 \\ 0 & \sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix}. \quad (10)$$

IV. Linear and Anti-Unitary Rotation Matrices

Which physical transformations exist to implement the six transformations R_1 to R_6? The natural candidates are the linear polarization transformations describing a polarization rotation, which will be called T_1 and a variable birefringence retardation T_2 (the latter actually describes a relative time shift, but is usually associated with polarization rather than with time). They can be related to the previously defined rotation matrices as

$$T_1(\alpha) = R_3(\alpha), \text{ and} \quad (11)$$

$$T_2(\alpha) = R_1(\alpha). \quad (12)$$

In addition one can shift the entire field, both polarizations, in time as

$$T_3(\alpha) = R_4(\alpha). \quad (13)$$

Using these three well-known physical operations we can realize the operations R_1, R_2, R_3, and R_4, from (11-13) and the matrix product

$$R_2 = T_2(\pi/4)T_1(\alpha)T_2(-\pi/4). \quad (14)$$

However, no combination of the three transformations T_1, T_2, and T_3 will result in the vector rotations R_5 or R_6. To be able to express these matrices as a combination of "elementary" physical transformations we need to introduce the transformation matrix

$$T_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (15)$$

This matrix leaves the a mode invariant and shifts the sign of $\hat{b}_2$ without shifting the sign of $\hat{b}_1$. It thus represent the conjugation operator † on the b mode. On the classical, complex field-vector $\overline{E}_y = \text{Re}(e_y) + i\text{Im}(e_y)$ it corresponds to $\overline{E}_y^*$. This is an anti-unitary operation [20] and it can therefore not be implemented without "cost" as the previous transformations T_1 - T_3. Quantum mechanically speaking, there is no Hamiltonian that will result in the transformation T_4, as any transformation originating from a Hamiltonian generator will result in a unitary transformation. However, with T_4 in our possession we find that

$$R_5(\alpha) = MT_1(-\alpha)M, \text{ and} \quad (16)$$

$$R_6(\alpha) = T_4T_1(\alpha)T_4, \quad (17)$$

where $M = T_4T_3(\pi/4)T_2(\pi/4)$ is a transformation that we will discuss in the next section. Thus, the transformations R_5 and R_6 are not unphysical in the strictest sense of the word, but they cannot be implemented on the same footing and without cost as can the rotations R_1 - R_4.

Alternatively this can be understood from the uncertainty relation (3), which can be written for both polarizations in 4d matrix form as

$$\hat{E}^t\lambda_1\hat{E} = i. \quad (18)$$

For an arbitrary 4d rotation operator R, the condition to be "physical" is to satisfy (18), which is $\hat{E}^tR^t\lambda_1R\hat{E} = i = \hat{E}^t\lambda_1\hat{E}$. The implication is, for arbitrary vectors $\hat{E}$, that $[R, \lambda_1] = 0$. The reader may verify that rotations R_1-R_4 do indeed commute with λ_1, whereas R_5 and R_6 do not. This also shows that operations that may seem symmetric in the 4d signal space may differ quantum mechanically, as the polarization and quadrature degrees of freedom are not equivalent with respect to the uncertainty relation.

V. Homodyning of Non-Commuting Quadratures

In the previous section we saw that in order to achieve the vector rotations R_5 and R_6 we need to employ an anti-unitary operation. For example, the transformation

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{19}$$

simply swaps $\hat{b}_1$ and $\hat{b}_2$. In the time domain it means we should "advance" the quadrature (or sine-) field in time by half a period in relation to the in-phase (or cosine-) field. This is clearly an operation that will require a measurement of the two quadratures separately, and then a re-modulation of the time-swapped quadratures onto a carrier signal. However, since $\hat{b}_1$ and $\hat{b}_2$ do not commute, we cannot measure them both without penalty. The simplest, (and in fact optimal, from an added noise point of view [16]) is to split the y-polarized field in two equal parts by the means of a 50:50 beam-splitter. Quantum mechanically this transformation will result in the new operators $\hat{c}$ and $\hat{d}$ where

$$\hat{c} = \; (\hat{b} + \hat{v})/\sqrt{2}, \text{ and} \tag{20}$$

$$\hat{d} = \; (\hat{b} - \hat{v})/\sqrt{2}. \tag{21}$$

The operator $\hat{v}$ is the annihilation operator associated to the vacuum (or zero-point) field entering through the unused, open port of the 50:50 beam-splitter. This operator and its associated quadrature amplitude operators satisfy similar commutation relations as do $\hat{a}$ and $\hat{b}$ and their associated quadrature operators. If the quadrature noise of the state in the b mode is much larger than the vacuum field noise level, then one may neglect the contribution from the vacuum fields, and then one ends up in a purely classical description where the quadratures can be measured simultaneously, with, in theory, arbitrary accuracy and no added noise. However, if (as in practice) the b mode is in a coherent state, then the quadrature noises of $\hat{b}$ and $\hat{v}$ are equally large, and the added quantum noise will degrade the SNR of the c and d modes by a factor 1/2 relative to the SNR of the b mode.

Having divided the b mode into two halves, we can in principle make a perfect measurement of $\hat{c}_1$ on one half and of $\hat{d}_2$ on the other half to get good estimates of $\hat{b}_1$ and $\hat{b}_2$, respectively. This is done by making a balanced homodyne measurement with a local oscillator whose amplitude is much larger than the measured signal's [16].

VI. Implementation of "Unphysical" Rotation Matrices

Using (16) and the information in the previous section we are ready to draw a schematic setup that will implement the transformation $R_5(\alpha)$. The setup is shown in Fig. 1. In the leftmost polarizing beam-splitter (PBS) the y-polarized b mode is separated from the x-polarized a mode. The b mode is subsequently split into two equal halves, resulting in the two modes described by Eqns. (20) and (21). The in-phase and quadrature-phase amplitudes of these two modes are measured by two balanced homodyne receivers, respectively, and the measured signals $\hat{c}_1$ and of $\hat{d}_2$ are used to modulate a "new" optical mode with the signal quadratures swapped with respect to the mode b. Since we only measure one quadrature of each mode, the measurement

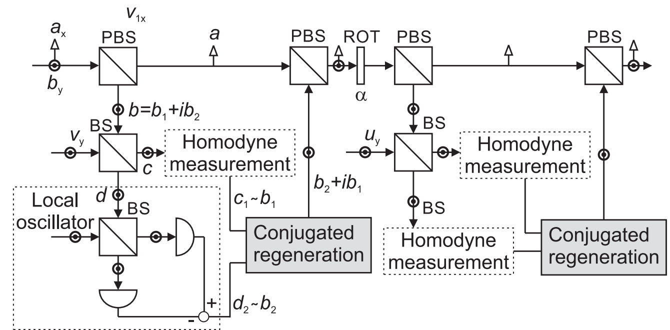

Fig. 1. A schematic setup generating the $R_5(\alpha)$ transformation. Four homodyne measurements need to be made of the appropriate quadrature amplitudes. In the lower left one such measurement is outlined. The other three of these measurements are only indicated by a dashed box. Where the y-polarized vacuum fields v and u enter are shown in the figure. Vacuum fields not contributing to the final output are not indicated. ROT denotes a polarization rotator, BS a beam splitter, and PBS a polarizing beam splitter.

need not introduce any additional noise into $\hat{c}_1$ or $\hat{d}_2$. As can be seen from Fig. 1, the operations up to this point corresponds to the matrix M in (19) acting on the initial four-vector. The y-polarized field entering the second PBS from the left is indeed $\hat{b}_2 + i\hat{b}_1$ with some added noise.

This y-polarized field is recombined with the x-polarized a mode in the second PBS from the left. The polarization of this recombined field is subsequently rotated the angle α. The polarization rotated field again is divided in its x- and y-polarized components by a third PBS. The y-polarized component is subsequently split in a 50:50 beam splitter, and a similar sequence of homodyning, quadrature amplitude swapping, and re-modulation as was described above, corresponding to the matrix M, is again employed. Finally the two polarizations are recombined in the rightmost PBS. The resulting signal vector is

$$\begin{pmatrix} \cos(\alpha)\hat{a}_1 - \sin(\alpha)(\hat{b}_2 - \hat{v}_2) \\ \cos(\alpha)\hat{a}_2 - \sin(\alpha)(\hat{b}_1 + \hat{v}_1) \\ \sin(\alpha)\hat{a}_2 + \cos(\alpha)(\hat{b}_1 + \hat{v}_1) - \hat{u}_2 \\ \sin(\alpha)\hat{a}_1 + \cos(\alpha)(\hat{b}_2 - \hat{v}_2) - \hat{u}_1 \end{pmatrix}. \tag{22}$$

If we assume that the input fields $\hat{a}$ and $\hat{b}$ are in coherent states, so that the quadrature amplitude variances $(\Delta\hat{a}_1)^2 = (\Delta\hat{a}_2)^2 = (\Delta\hat{b}_1)^2 = (\Delta\hat{b}_2)^2 = 1/4$, then the input and output SNRs are given in Table I.

It is seen that overall, noise is added and the SNR is degraded. If, e.g., α is $\pi/2$ or $3\pi/2$, so that both the polarizations and the associated quadrature amplitudes are swapped, the SNR in each channel suffers a 3 dB penalty. We also see that the four channels suffer differently from the added noise.

An alternative way of implementing the transformation $R_5(\alpha)$ is to split both the a and the b mode into equal parts and then making separate, homodyne measurements of the four signals $\hat{a}_1$, $\hat{a}_2$, $\hat{b}_1$, and $\hat{b}_2$. This will decrease the SNR of each of these signals with the factor 1/2 as is the consequence of the splitting transformation in (20) and (21) and the corresponding relations for the split a mode. The signals, that after the

TABLE I
INPUT AND OUTPUT SNR FOR TRANSFORMATION $R_5(\alpha)$ IMPLEMENTED AS IN FIG. 1

Component	SNR in	SNR out
$\mathrm{Re}(e_x)$	$4\langle\hat{a}_1\rangle^2$	$\frac{4\langle\cos(\alpha)\hat{a}_1-\sin(\alpha)\hat{b}_2\rangle^2}{1+\sin^2(\alpha)}$
$\mathrm{Im}(e_x)$	$4\langle\hat{a}_2\rangle^2$	$\frac{4\langle\cos(\alpha)\hat{a}_2-\sin(\alpha)\hat{b}_1\rangle^2}{1+\sin^2(\alpha)}$
$\mathrm{Re}(e_y)$	$4\langle\hat{b}_1\rangle^2$	$\frac{4\langle\sin(\alpha)\hat{a}_2+\cos(\alpha)\hat{b}_1\rangle^2}{2+\cos^2(\alpha)}$
$\mathrm{Im}(e_y)$	$4\langle\hat{b}_2\rangle^2$	$\frac{4\langle\sin(\alpha)\hat{a}_1+\cos(\alpha)\hat{b}_2\rangle^2}{2+\cos^2(\alpha)}$

TABLE II
INPUT AND OUTPUT SNR FOR TRANSFORMATION $R_5(\alpha)$ IMPLEMENTED BY HOMODYNING OF THE POLARIZATION SEPARATED, AND SUBSEQUENTLY EQUALLY SPLIT FIELDS

Component	SNR in	SNR out
$\mathrm{Re}(e_x)$	$4\langle\hat{a}_1\rangle^2$	$2\langle\cos(\alpha)\hat{a}_1-\sin(\alpha)\hat{b}_2\rangle^2$
$\mathrm{Im}(e_x)$	$4\langle\hat{a}_2\rangle^2$	$2\langle\cos(\alpha)\hat{a}_2-\sin(\alpha)\hat{b}_1\rangle^2$
$\mathrm{Re}(e_y)$	$4\langle\hat{b}_1\rangle^2$	$2\langle\sin(\alpha)\hat{a}_2+\cos(\alpha)\hat{b}_1\rangle^2$
$\mathrm{Im}(e_y)$	$4\langle\hat{b}_2\rangle^2$	$2\langle\sin(\alpha)\hat{a}_1+\cos(\alpha)\hat{b}_2\rangle^2$

TABLE III
INPUT AND OUTPUT SNR FOR TRANSFORMATION $R_6(\alpha)$ IMPLEMENTED BY HOMODYNING OF OF THE POLARIZATION SEPARATED, AND SUBSEQUENTLY EQUALLY SPLIT FIELDS

Component	SNR in	SNR out
$\mathrm{Re}(e_x)$	$4\langle\hat{a}_1\rangle^2$	$2\langle\cos(\alpha)\hat{a}_1+\sin(\alpha)\hat{b}_1\rangle^2$
$\mathrm{Im}(e_x)$	$4\langle\hat{a}_2\rangle^2$	$2\langle\cos(\alpha)\hat{a}_2-\sin(\alpha)\hat{b}_2\rangle^2$
$\mathrm{Re}(e_y)$	$4\langle\hat{b}_1\rangle^2$	$2\langle-\sin(\alpha)\hat{a}_1+\cos(\alpha)\hat{b}_1\rangle^2$
$\mathrm{Im}(e_y)$	$4\langle\hat{b}_2\rangle^2$	$2\langle\sin(\alpha)\hat{a}_2+\cos(\alpha)\hat{b}_2\rangle^2$

measurement are classical, can then be re-modulated in any desired superposition onto a two coherent states, one x-polarized and one y-polarized, for example in the combination given by $R_5(\alpha)$. The two modulated fields can subsequently be made to propagate into the same transverse and longitudinal mode by merging the states in a PBS. The corresponding SNR table in this case is Table II.

Such an implementation of $R_5(\alpha)$ distributes the unavoidable, added noise due to the anti-unitarity of $R_5(\alpha)$ in a symmetric fashion by treating all four signal vector components in a similar manner.

Yet another way of implementing the rotation $R_5(\alpha)$ is to amplify the classical signals from leftmost pair of homodyne measurements in Fig. 1. If the (power) gain $G \gg 1$, the optical field leaving the leftmost conjugate regeneration source has a quadrature amplitude variance $G/4$ that is much above the quantum limit 1/4, then it can subsequently be treated as a classical field.

In this case we also must amplify the a mode by the (power) gain G in order to create the proper superposition after the half-wave plate. However, linear, phase insensitive amplification also results in added noise as the proper transformation law to satisfy the bosonic commutator $[\hat{e}^\dagger, \hat{e}] = 1$ is

$$\hat{e} = \sqrt{G}\hat{a} + \sqrt{G-1}\hat{w}^\dagger, \tag{23}$$

where $\hat{a}$ is the input mode, $\hat{e}$ the output mode, $\hat{w}$ is a amplifier internal mode, nominally in a vacuum state, and G is the (power) gain of the amplifier [14]. Thus, if $G \gg 1$, the SNR of the amplified mode e is effectively reduced by the factor 1/2 as compared with the a mode SNR, just as the SNR of the b mode was reduced by the same factor by the simultaneous measurement of $\hat{b}_1$ and $\hat{b}_2$. However, the transformations including and following the half-wave plate in Fig. 1 are now operating on what can be considered two classical fields (the added quantum noise can be neglected), so the output SNR table for such an implementation is identical to Table II.

In a similar manner, the transformation $R_6(\alpha)$ can be implemented. If the implementation treats the two polarizations in a symmetric fashion, again the SNR table of the transformation is given by Table III.

VII. AN EXAMPLE

In [7] it was pointed out that information encoded in the polarization degree of freedom in a coherent communication system is generally easier to track than the information encoded in the quadrature amplitudes of a fixed polarization. The reason is due to differences between the noise spectra of these degrees of freedom. While the dominant noise of the polarization is at low frequencies, needing polarization locking loops with a time response of milliseconds, the higher frequency noise in the quadrature amplitudes require phase-lock loops with a response in the microsecond regime. Thus it can be advantageous to convert signals with much of the information in the quadrature amplitude domain to a signal that has more of the information to the polarization domain. As an example, look at the 3 bit, polarization switched QPSK signals

$$\{\{\pm 1,0,0,0\}^t, \{0,\pm 1,0,0\}^t, \{0,0,\pm 1,0\}^t, \{0,0,0,\pm 1\}^t\} \tag{24}$$

where the superscript t indicates transpose. These signals represent signals that use two orthogonal polarizations and a fourfold, symmetric phase degeneracy. (In each of the polarizations the amplitudes $\pm E$ and $\pm iE$ are used.) To convert some of the information into the polarization degree of freedom the transformation $R_6(\pi/8)$ can be used. The transformed signals become

$$\{\pm\{c,0,s,0\}^t, \pm\{0,c,0,-s\}^t, \pm\{-s,0,c,0\}^t, \pm\{0,s,0,c\}^t\} \tag{25}$$

where $c = \cos(\pi/8)$ and $s = \sin(\pi/8)$. Here, four polarizations are used, and for each of the four polarization states an antipodal phase modulation is used. All four polarizations states cannot be orthogonal, but in signal space the distance between the signals remain invariant since all signals experience the same transformation. A result of the transformation is that the SNR of all the signals will have decreased by 3 dB (in the symmetric case and if the signals are coherent states). However, this may be a price one is willing to pay since the tracking of the polarization drift is simpler than tracking of the phase drift. The noise added by the transformation from one modulation constellation to the other is white, so it does not contribute to neither the polarization drift nor to the phase drift.

VIII. Conclusion

If a coherent communication system using phase- and polarization multiplexing is treated by the real, four-component vector formalism suggested in [7] it has been shown that any vector (and thus signal) transformation can be described by six matrices, e.g., chosen as (5)-(10) [7]. Two of these transformations (9) and (10) have been labeled "unphysical", a designation that is correct if only linear polarization transformations and phase-shifts are considered. In this paper we have discussed why these two transformations are different from the other four, and shown that they involve anti-unitary transformations. Such transformations are not prohibited *per se*, but they can only be performed at the cost of a decreased SNR [21].

Perhaps the most important observation we make is to note that while operators operating on orthogonal polarizations of the field commute, the quadrature amplitude operators do not. Thus, only operations common to the two quadrature amplitudes in one polarization mode, such as a phase shift affecting both quadratures equally, can be done without adding noise. Swapping the quadrature amplitudes, i.e., the transformation $\hat{a}_1 \leftrightarrow \hat{a}_2$, is not unitary and can only be done at an SNR cost. Here, the classical description of the fields, that assume that since the two quadrature are signal orthogonal they can be treated on the same footing as orthogonal polarization signals, differ from the quantum description. If the noise in the signals is much larger than the minimum noise dictated by quantum mechanics, then the classical description gives an accurate prediction of how the signals can be manipulated and measured. However, for signals in a coherent state, whose noise is at the (standard) quantum limit, a quantum description must be used to accurately assessing and predicting the transformation and measurement of the signals.

References

[1] F. Derr, "Coherent optical QPSK intradyne system: Concept and digital receiver realization," *J. Lightw. Technol.*, vol. 10, no. 9, pp. 1290–1296, Sep. 1992.

[2] H. Sun, K.-T. Wu, and K. Roberts, "Real-time measurements of a 40 Gb/s coherent system," *Opt. Express*, vol. 16, no. 2, pp. 873–879, 2008.

[3] S. Betti, F. Curti, G. De Marchis, and E. Iannone, "A novel multilevel coherent optical system: 4-quadrature signaling," *J. Lightw. Technol.*, vol. 9, no. 4, pp. 514–523, Apr. 1991.

[4] E. Agrell and M. Karlsson, "Power-efficient modulation formats in coherent transmission systems," *J. Lightw. Technol.*, vol. 27, pp. 5115–5126, Nov. 2009.

[5] R. C. Jones, "A new calculus for the treatment of optical systems I. Description and discussion of the calculus," *J. Opt. Soc. Amer.*, vol. 31, no. 7, pp. 488–493, 1941.

[6] H. Hurwitz Jr and R. C. Jones, "A new calculus for the treatment of optical systems II. Proof of three equivalence theorems," *J. Opt. Soc. Amer.*, vol. 31, no. 7, pp. 493–495, 1941.

[7] M. Karlsson, "Four-dimensional rotations in coherent optical communications," *J. Lightw. Technol.*, vol. 32, no. 6, pp. 1246–1257, Mar. 2014.

[8] D. E. Crivelli *et al.*, "Architecture of a single-chip 50 Gb/s DP-QPSK/BPSK transceiver with electronic dispersion compensation for coherent optical channels," *IEEE Trans. Circuits Syst.*, vol. 61, no. 4, pp. 1012–1025, Apr. 2014.

[9] H. Louchet, A. Richter, and K. Kuzmin, "Joint carrier-phase and polarization rotation recovery for arbitrary signal constellations," *IEEE Photonics Technol. Lett.*, vol. 26, no. 9, pp. 922–924, May 2014.

[10] C. B. Czegledi, E. Agrell, and M. Karlsson, "Symbol-by-symbol joint polarization and phase tracking in coherent receivers," in *Proc. Opt. Fiber Commun. Conf. Exhib.*, 2015, pp. W1E–3.

[11] W. K. Wootters and W. H. Zurek, "A single quantum cannot be cloned," *Nature*, vol. 299, no. 5886, pp. 802–803, Oct. 1982. [Online]. Available: http://dx.doi.org/10.1038/299802a0

[12] D. Dieks, "Communication by EPR devices," *Phys. Lett. A*, vol. 92, no. 6, pp. 271–272, 1982. [Online]. Available: http://www.sciencedirect.com/science/article/pii/0375960182900846

[13] H. P. Yuen, "Amplification of quantum states and noiseless photon amplifiers," *Phys. Lett. A*, vol. 113, no. 8, pp. 405–407, 1986. [Online]. Available: http://www.sciencedirect.com/science/article/pii/0375960186906602

[14] C. M. Caves, "Quantum limits on noise in linear amplifiers," *Phys. Rev. D*, vol. 26, no. 8, pp. 1817–1839, Oct. 1982.

[15] V. Bužek and M. Hillery, "Quantum copying: Beyond the no-cloning theorem," *Phys. Rev. A*, vol. 54, pp. 1844–1852, Sep. 1996. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevA.54.1844

[16] Y. Yamamoto and H. A. Haus, "Preparation, measurement and information capacity of optical quantum states," *Rev. Modern Phys.*, vol. 58, pp. 1001–1020, Oct. 1986. [Online]. Available: http://link.aps.org/doi/10.1103/RevModPhys.58.1001

[17] H. P. Yuen and V. W. S. Chan, "Noise in homodyne and heterodyne detection," *Opt. Lett.*, vol. 8, no. 3, pp. 177–179, Mar. 1983. [Online]. Available: http://ol.osa.org/abstract.cfm?URI=ol-8-3-177

[18] B. L. Schumaker, "Noise in homodyne detection," *Opt. Lett.*, vol. 9, no. 5, pp. 189–191, May 1984. [Online]. Available: http://ol.osa.org/abstract.cfm?URI=ol-9-5-189

[19] M. O. Scully and M. S. Zubairy, *Quantum Optics*. Cambridge, U.K.: Cambridge Univ. Press, 1997.

[20] E. Wigner and J. Griffin, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Series Pure and applied Physics). New York, NY, USA: Academic, 1959. [Online]. Available: https://books.google.se/books?id=IOJEAAAAIAAJ

[21] N. J. Cerf and S. Iblisdir, "Phase conjugation of continuous quantum variables," *Phys. Rev. A*, vol. 64, Aug. 2001, Art. no. 032307. [Online]. Available: http://link.aps.org/doi/10.1103/PhysRevA.64.032307

Gunnar Björk received the M.Sc. degree in engineering physics and the Ph.D. degree in electrical engineering in from the Royal Institute of Technology (KTH), Stockholm, Sweden, in 1984 and 1988, respectively. He is currently a Full Professor at KTH, Stockholm, Sweden. He has previously held positions, or been a visitor, at NTT Basic Research Laboratories, Musashino, Japan, Stanford University, USA, Universidad Complutense, Madrid, Spain, Nihon University, Tokyo, Japan, and the Max Planck Institute for the Science of Light at Erlangen, Germany. His main research interest include quantum optics and quantum information. He is a European Physical Society Fellow.

Katarina Stensson was born in Örebro, Sweden in 1988. She received the M.Sc. degree in engineering physics in 2014, from the Royal Institute of Technology (KTH), Stockholm, Sweden. Her master thesis was on quantum optics in two-dimensional nonlinear QPM crystals. She is currently working toward the Ph.D. degree in optics and photonics at KTH, under the supervision of Prof. Gunnar Björk. Her main research interests include quantum optics and quantum information and communication.

Magnus Karlsson (SM'13) received the Ph.D. degree from the Chalmers University of Technology, Gothenburg, Sweden, in 1994. Since 1995, he has been with the Photonics Laboratory at Chalmers, first as an Assistant Professor and since 2003 as a Professor in photonics. He has authored or coauthored around 300 scientific journal and conference contributions in the areas of nonlinear optics and fiber optic transmission.

He serves in the technical program committees for the Optical Fiber Communication Conference, the European Conference of Optical Communication, and for Globecom and ICC. He is an Associate Editor for *Optics Express* since 2010 and a Deputy Editor since 2015. He was appointed a Fellow of the Optical Society of America in 2012. His research interests include linear and nonlinear fiber transmission effects and multilevel modulation in optical communication systems.